G53KRR 2014-2015 handout on description logic

OWL Web Ontology Language - W3C standard, extends most description logics and has slightly different terminology (based on RDF rather than description logic semantics. OWL DL based on DL).

Reading:
The Description Logic Handbook. Franz Baader, Diego Calvanese, Deborah L.McGuinness, Daniele Nardi, and Peter F. Patel-Schneider, editors. Cambridge University Press, 2003. ISBN 0-521-78176-0.

A good on-line course: `http://www.inf.unibz.it/%7Efranconi/dl/course/`

Basic idea description logics talk about relationships between *concepts* (noun phrases). There are many different description logics.

Description logics $\mathcal{EL}$, $\mathcal{ALC}$, $\mathcal{ALCQ}$:
Concept descriptions in $\mathcal{EL}$ are formed using the following syntax rule:

$$C, D \longrightarrow A \mid \top \mid \bot \mid C \sqcap D \mid \exists R.C,$$

where A is an atomic concept, R is an atomic role, n is a non-negative natural number.

Concept descriptions in $\mathcal{ALC}$ are formed using the following syntax rule:

$$C, D \longrightarrow A \mid \top \mid \bot \mid C \sqcap D \mid \neg C \mid \forall R.C,$$

where A is an atomic concept, R is an atomic role.
$C \sqcup D \equiv \neg(\neg C \sqcap \neg D)$. $\exists R.C \equiv \neg(\forall R.(\neg C))$

Concept descriptions in $\mathcal{ALCQ}$ are formed using the following syntax rule:

$$C, D \longrightarrow A \mid \top \mid \bot \mid C \sqcap D \mid \neg C \mid \forall R.C \mid \exists^{>n} R.C \mid \exists^{=n} R.C \mid \exists^{<n} R.C,$$

where A is an atomic concept, R is an atomic role, n is a non-negative natural number.
$\exists^{>n} R \equiv \exists^{>n} R.\top$. $\exists^{\geq n} R \equiv (\exists^{>n} R) \sqcup (\exists^{=n} R)$.

A sentence ϕ is defined as follows:

$$\phi := C \sqsubseteq D \mid C(a) \mid R(a, b),$$

where C, D are concept descriptions, a, b are individual names, R is an atomic role.
$C \equiv D$ is short for $C \sqsubseteq D$ and $D \sqsubseteq C$.

An interpretation $(\Delta, \mathcal{I})$ consists of a non-empty set Δ as the interpretation domain and an interpretation function $\mathcal{I}$, which assigns every atomic concept A to a set $A^{\mathcal{I}} \subseteq \Delta$, every atomic role R to a binary relation $R^{\mathcal{I}} \subseteq \Delta \times \Delta$, and every individual name a to an element $a^{\mathcal{I}} \in \Delta$. The interpretation function is extended to concept descriptions as follows:

1. $\top^{\mathcal{I}} = \Delta$, $\bot^{\mathcal{I}} = \emptyset$;
2. $(C \sqcap D)^{\mathcal{I}} = C^{\mathcal{I}} \cap D^{\mathcal{I}}$;
3. $(\neg C)^{\mathcal{I}} = \Delta \setminus C^{\mathcal{I}}$;
4. $(\forall R.C)^{\mathcal{I}} = \{a \in \Delta \mid \forall b.\ (a, b) \in R^{\mathcal{I}} \to b \in C^{\mathcal{I}}\}$;
5. $(\exists^{>n} R.C)^{\mathcal{I}} = \{a \in \Delta \mid |\{b \in \Delta \mid (a, b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}| > n\}$;
6. $(\exists^{=n} R.C)^{\mathcal{I}} = \{a \in \Delta \mid |\{b \in \Delta \mid (a, b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}| = n\}$;
7. $(\exists^{<n} R.C)^{\mathcal{I}} = \{a \in \Delta \mid |\{b \in \Delta \mid (a, b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}| < n\}$.

The truth conditions for sentences are as follows:

1. $(\Delta, \mathcal{I}) \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$;
2. $(\Delta, \mathcal{I}) \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$;
3. $(\Delta, \mathcal{I}) \models R(a, b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$.

A description logic knowledge base is a set of sentences.

TBox and ABox A description logic knowledge base is usually split into terminological part or TBox which describes general relationships between concepts, e.g. $Surgeon \sqsubseteq Doctor$, and assertions about individuals or ABox (e.g. $Doctor(mary)$).

Reasoning Entailment is defined exactly like in FOL: a set of sentences Γ entails a sentence ϕ (in symbols $\Gamma \models \phi$) if and only if ϕ is true in every interpretation where all of the sentences in Γ are true.

$\mathcal{ALC}$ is a proper fragment of first order logic. Reasoning in $\mathcal{ALC}$ it is decidable (it is decidable whether a sentence is satisfiable, or whether a finite set of sentences entails another sentence; however algorithms for checking this take exponential time).

Example of a description logic where reasoning is very efficient: $\mathcal{EL}$ only has $\sqcap$ and $\exists.R$ as concept constructors. Reasoning not just decidable, but very efficient (polynomial time algorithm for checking subsumption of concepts).

Other features used to define more expressive description logics: functional roles (for example, to say that only one object can be connected by an Age role), cardinality restrictions on the number of objects connected by a role, ability to say that roles are transitive, reflexive, express inclusion relation between roles. Some very expressive description logics are undecidable.

G53KRR 2012, Q6 This question is on description logic. A summary of its syntax and semantics is given after this question.

1. Express the following sentence in description logic:

   (a) Bob is a student.
   (b) Bob and David are friends.
   (c) Every child has a mother.
   (d) Bob has more than 2 sisters which are students.
   (e) Bob has no brothers.
   (f) Bob has no more than 2 sisters but no brothers
   (g) Each of Bob's sisters has 2 brothers.
   (h) All of Bob's friends are students.
   (i) Some of Bob's friends are not students.

   Answer.

   (a) $Student(bob)$
   (b) $hasFriend(bob, david)$ or $hasFriend(david, bob)$
   (c) $Child \sqsubseteq \exists hasMother.\top$;
   (d) $(\exists^{>2} hasSister.Student)(bob)$;
   (e) $(\forall hasBrother.\bot)(bob)$;
   (f) $(\exists^{\leq 2} hasSister \sqcap \forall hasBrother.\bot)(bob)$;
   (g) $(\forall hasSister.(\exists^{\geq 2} hasBrother))(bob)$;
   (h) $(\forall hasFriend.Student)(bob)$;

(i) $(\exists hasFriend.(\neg Student))(bob)$

Other equivalent DL expressions are also correct.

2. Consider the following interpretation (D, I): $D = \{d_1, d_2, d_3\}$, $R^I = \{\langle d_1, d_2\rangle, \langle d_1, d_3\rangle\}$, $a^I = d_1$ (a is a constant), $C^I = \{d_2, d_3\}$ (C is an atomic concept). Which of the following sentences are true in this interpretation and why?

   (a) $(\forall R.C)(a)$

   (b) $(\exists^{\geq 1} R)(a)$

   (c) $(\exists^{\geq 2} R) \sqsubseteq (\exists^{\geq 1} R)$

   Answer.

   (a) True. $a^I = d_1$. Since $R^I = \{\langle d_1, d_2\rangle, \langle d_1, d_3\rangle\}$, $d_2 \in C^I$ and $d_3 \in C^I$, we have $\forall x.\ (d_1, x) \in R^I \to x \in C^I$. By the definition of $(\forall R.C)^I$, $d_1 \in (\forall R.C)^I$. This is, $a^I \in (\forall R.C)^I$. By the truth condition of $(\forall R.C)(a)$, $(D, I) \models (\forall R.C)(a)$.

   (b) True. $a^I = d_1$. $|\{x \in D \mid (d_1, x) \in R^I\}| = 2$, since x could be d_2 or d_3.

   (c) True. As $(\exists^{\geq 2} R)^I \subseteq (\exists^{\geq 1} R)^I$, $(\exists^{\geq 2} R) \sqsubseteq (\exists^{\geq 1} R)$ is a valid sentence, which is true in every interpretation.

Summary of the syntax and semantics of the DL: Concept descriptions in the DL are formed using the following syntax rule:

$$C, D \longrightarrow A \mid \top \mid \bot \mid C \sqcap D \mid \neg C \mid \forall R.C \mid \exists^{>n} R.C \mid \exists^{=n} R.C \mid \exists^{<n} R.C,$$

where A is an atomic concept, R is an atomic role, n is a non-negative natural number. $C \sqcup D \equiv \neg(\neg C \sqcap \neg D)$. $\exists R.C \equiv \neg(\forall R.(\neg C))$. $\exists^{>n} R \equiv \exists^{>n} R.\top$. $\exists^{\geq n} R \equiv (\exists^{>n} R) \sqcup (\exists^{=n} R)$.

A sentence ϕ is defined as follows:

$$\phi := C \sqsubseteq D \mid C(a) \mid R(a, b),$$

where C, D are concept descriptions, a, b are individual names, R is an atomic role.

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- $(\forall R.C)^{\mathcal{I}} = \{a \in \Delta \mid \forall b.\ (a, b) \in R^{\mathcal{I}} \to b \in C^{\mathcal{I}}\}$;
- $(\exists^{>n} R.C)^{\mathcal{I}} = \{a \in \Delta \mid |\{b \in \Delta \mid (a, b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}| > n\}$;
- $(\exists^{=n} R.C)^{\mathcal{I}} = \{a \in \Delta \mid |\{b \in \Delta \mid (a, b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}| = n\}$.
- $(\exists^{<n} R.C)^{\mathcal{I}} = \{a \in \Delta \mid |\{b \in \Delta \mid (a, b) \in R^{\mathcal{I}} \wedge b \in C^{\mathcal{I}}\}| < n\}$;
- $(\exists^{>n} R)^{\mathcal{I}} = \{a \in \Delta \mid |\{b \in \Delta \mid (a, b) \in R^{\mathcal{I}}\}| > n\}$;
- $(\exists^{=n} R)^{\mathcal{I}} = \{a \in \Delta \mid |\{b \in \Delta \mid (a, b) \in R^{\mathcal{I}}\}| = n\}$;
- $(\exists^{<n} R)^{\mathcal{I}} = \{a \in \Delta \mid |\{b \in \Delta \mid (a, b) \in R^{\mathcal{I}}\}| < n\}$.

The truth conditions for sentences are as follows:

- $(\Delta, \mathcal{I}) \models C \sqsubseteq D$ iff $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$;
- $(\Delta, \mathcal{I}) \models C(a)$ iff $a^{\mathcal{I}} \in C^{\mathcal{I}}$;
- $(\Delta, \mathcal{I}) \models R(a, b)$ iff $(a^{\mathcal{I}}, b^{\mathcal{I}}) \in R^{\mathcal{I}}$.