Fixed-Point Characterisation of Winning Strategies in Impartial Games

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Abstract

We use fixed-point calculus to characterise winning strategies in impartial, two-person games. A byproduct is the fixed-point characterisation of winning, losing and stalemate positions. We expect the results to be most useful in teaching calculational reasoning about least and greatest fixed points.

An abridged version of this paper (omitting some proofs) has been submitted for publication.

Game theory [BCG82] is an active area of research for computing scientists. For example, it is a frutiful source of examples illustating complexity theory, and it is also used as the basis for the semantics of model checking. Our interest in the area is as a test case for the use of formalisms for the constructive derivation of algorithms. Game theory is well-suited to our goals because it is about constructing winning strategies. Moreover, examples of games are easy to explain to students, they carry no theoretical overhead, and motivation is for free.

In the study of games, as in the book "Winning Ways" [BCG82], a basic assumption is that all games are terminating. Only limited attention has been paid to games where non-termination is possible (so-called "loopy" games [BCG82, chapter 12]). In this paper, we study impartial two-person games, in which non-termination (caused by loops or infinite game graphs) is a possibility. We show how to characterise winning, losing and stalemate positions in terms of least and greatest fixed points of conjugate predicate transformers.

The division of *positions* into winning, losing and stalemate positions is well-known (see, for example, [SS93] or [BCG82]) . The contribution of this paper is to focus on winning *strategies*; we formalise their construction in point-free relation algebra. A byproduct is the fixed-point characterisation of the different types of position.

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1 Impartial Two-Person Games

An *impartial, two-person game* is defined by a binary relation, denoted here by M. Elements of the domain of M are called *positions*; pairs of positions related by M are called *moves*. We make no restrictions on M; in particular, the set of positions may be infinite, and M is not required to be well-founded.

Figure 1 is an example of a (non-well-founded) move relation. The positions are the nodes in the figure, and moves are indicated by arrows. We call it the *lollipop game*.

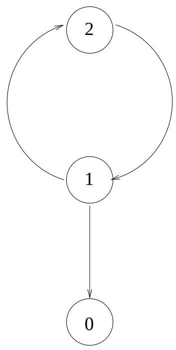

Figure 1: The lollipop game

A game is started in a given position. Each player takes it in turn to move; in position s, a move is to a position t such that $s\,M\,t$. The game ends when a player cannot move; the player whose turn it is to move then loses.

In fig. 1, the game ends in position 0. In position 1, there is a choice of two moves, either to position 0 or to position 2. In position 2, there is no choice of move; the only move that can be made is to position 1. The move relation is not well-founded because it is possible for a game to continue indefinitely, by cycling between positions 1 and 2.

Allowing the move relation to be non-well-founded introduces additional difficulties in the development of the theory. For example, in traditional game theory, a fundamental element is the definition of an equivalence relation on games; that this relation is reflexive is established by a "tit-for-tat" winning strategy (the "Tweedledum and Tweedledee Argument" in [BCG82]). But, tit-for-tat is invalid in the case of non-well-founded game relations. One of our concerns is to identify precisely where the property of well-foundedness is vital to the development of the theory.

Throughout this paper, we use the Dijkstra-Scholten [DS90] notation for predicates and predicate transformers. In particular, we use square brackets to indicate that a predicate is true at all positions. For a given relation R, $dom.R$ and $rng.R$ are predicates characterising the domain and range of R, respectively. Formally, for all positions s,

$$dom.R.s \;\equiv\; \langle \exists t :: sRt \rangle$$

and, for all positions t,

$$\mathit{rng}.R.t \equiv \langle \exists s :: sRt \rangle \quad .$$

The composition of relations R and S is denoted by $R \bullet S$. Specifically, for all s and u,

$$s(R \bullet S)u \equiv \langle \exists t :: sRt \wedge tSu \rangle \quad .$$

2 Strategies and Position Predicates

2.1 Winning Strategies

A *winning* position in a game is one from which the first player has a strategy to choose moves that guarantee that the game ends, within a finite number of moves, on the second player's turn to move. A *losing* position is one from which there are only moves into winning positions. All other positions are *stalemate* positions.

Formally, a *winning strategy* for game M is a relation W on positions, with the following three properties:

(1) $W \subseteq M$,

(2) $W \bullet M$ is well-founded ,

(3) $\langle \forall t : \mathit{rng}.W.t : \langle \forall u : t\,M\,u : \mathit{dom}.W.u \rangle \rangle$.

In words, a winning strategy is a subset of the move relation —(1)— , such that repeatedly invoking the strategy, followed by making an arbitrary move, cannot continue indefinitely —(2)— , and, from every position in the range of the strategy, every move is to a position in the domain of the strategy —(3)— .

A position s is a *winning position* if s is in the domain of a winning strategy. A position t is a *losing position* if every move from t is to a winning position. A position that is not a winning position or a losing position is a *stalemate position*.

In the lollipop game (fig. 1), node 0 is a losing position, since it is vacuously true that every move from this position is to a winning position. Node 1 is a winning position — a winning strategy is to move from position 1 to position 0. (Formally, the winning strategy is the relation $\{(1,0)\}$.) Finally, node 2 is a losing position, since every move from this position is to node 1, which we have determined to be a winning position.

There are no stalemate positions in the lollipop game. For several entertaining, nontrivial examples of games with stalemate positions, see [BCG82, chapter 12].

2.2 Winning and Losing

From the definition of winning and losing positions, we can identify two properties of positions that they must satisfy. First, losing equivales every move is into a winning position: for all positions t,

$$(4) \qquad lose.t \;\equiv\; \langle \forall u : t\,M\,u : win.u \rangle \quad .$$

(This is by definition.) Second, from a winning position there is always a move into a losing position: for all positions s,

$$(5) \qquad win.s \;\Rightarrow\; \langle \exists t : s\,M\,t : lose.t \rangle \quad .$$

The proof is straightforward:

$$
\begin{array}{ll}
 & win.s \\
= & \{ \quad \text{definition of } win \quad \} \\
 & \langle \exists W : WinningStrategy.W : dom.W.s \rangle \\
= & \{ \quad \text{definition of } dom.W \quad \} \\
 & \langle \exists W : WinningStrategy.W : \langle \exists t :: s\,W\,t \rangle\rangle \\
= & \{ \quad \text{definition of } rng.W \text{ and (3)} \quad \} \\
 & \langle \exists W : WinningStrategy.W : \langle \exists t :: s\,W\,t \wedge \langle \forall u : t\,M\,u : dom.W.u \rangle\rangle\rangle \\
\Rightarrow & \{ \quad \text{calculus and (1)} \quad \} \\
 & \langle \exists t :: s\,M\,t \wedge \langle \exists W : WinningStrategy.W : \langle \forall u : t\,M\,u : dom.W.u \rangle\rangle\rangle \\
\Rightarrow & \{ \quad \text{(4) and calculus} \quad \} \\
 & \langle \exists t : s\,M\,t : lose.t \rangle \quad .
\end{array}
$$

Note that (5) is an implication, not an equivalence. (See the last two steps in the calculation.) Knowing that there is a move from position s to a losing position (i.e. $\langle \exists t : s\,M\,t : lose.t \rangle$) is not sufficient to construct a winning strategy with domain containing s. This is illustrated by the lollipop game (fig. 1). An ignorant player might repeatedly choose to move from node 1 (a winning position) to node 2 (a losing position) in the —mistaken— belief that a winning strategy is simply to always leave the opponent in a losing position.

The converse implication is nevertheless true. Demonstrating, by formal calculation, that this is the case is the driving force behind several of our calculations.

2.3 The Predicate Transformers $\mathcal{S}ome$ and $\mathcal{A}ll$

From (4), we abstract the predicate transformer $\mathcal{A}ll.R$, defined by

$$(6) \qquad \mathcal{A}ll.R.p.s \;\equiv\; \langle \forall t : sRt : p.t \rangle \quad ,$$

and, from (5), we abstract the predicate transformer $\mathcal{S}ome.R$, defined by

$$(7) \qquad \mathcal{S}ome.R.p.s \;\equiv\; \langle \exists t : sRt : p.t \rangle \quad .$$

In both definitions, R is a relation on positions, p is a predicate on positions, and s and t are positions.

The properties (3), (4) and (5) can be reformulated in a point-free form using these predicate transformers. Effective calculation is considerably enhanced by the convention of regarding a predicate on positions as a partial identity relation; the relation obtained by restricting the domain of a relation, R say, to positions satisfying a predicate, p say, is then simply the relation $p{\bullet}R$. Similarly, $R{\bullet}p$ is the relation obtained by restricting the range of relation R to positions satisfying predicate p. In this way, property (3) becomes:

$$(8) \qquad W \;=\; W{\bullet}\mathcal{A}ll.M.(dom.W) \quad .$$

Properties (4) and (5) become

$$(9) \qquad [lose \;\equiv\; \mathcal{A}ll.M.win] \quad ,$$

and

$$(10) \qquad [win \;\Rightarrow\; \mathcal{S}ome.M.lose] \quad .$$

We record some simple properties of $\mathcal{S}ome$ and $\mathcal{A}ll$ for later use. (The rules given here are used more than once. Other rules that are used once only are stated at the appropriate point in a calculation.) A property of $\mathcal{A}ll$ and rng is that, for all relations R and all predicates p,

$$(11) \qquad [\mathcal{A}ll.R.p \;\equiv\; \mathcal{A}ll.R.(rng.R \Rightarrow p)] \quad .$$

The function $\mathcal{S}ome$ is monotonic in its first argument. That is, for all relations R and S, and all predicates p,

$$(12) \qquad [\mathcal{S}ome.R.p \Rightarrow \mathcal{S}ome.S.p] \;\Leftarrow\; R \subseteq S \quad .$$

(The function $\mathcal{A}ll$ is *anti*monotonic in its first argument. But, we don't use this rule.) The predicate transformers $\mathcal{S}ome.R$ and $\mathcal{A}ll.R$ are monotonic. That is, for all relations R, and predicates p and q,

$$(13) \qquad [\mathcal{S}ome.R.p \;\Rightarrow\; \mathcal{S}ome.R.q] \;\Leftarrow\; [p \Rightarrow q] \quad ,$$

and

$$(14) \qquad [\mathit{All}.R.p \Rightarrow \mathit{All}.R.q] \;\Leftarrow\; [p \Rightarrow q] \quad .$$

Consequently, for all relations R, $\mathit{All}.R \circ \mathit{Some}.R$ and $\mathit{Some}.R \circ \mathit{All}.R$ are also monotonic. (We use "$\circ$" for the composition of functions.)

A crucial observation is that, for all relations R, $\mathit{Some}.R$ and $\mathit{All}.R$ are *conjugate* predicate transformers. That is,

$$(15) \qquad \mathit{All}.R \;=\; \neg \circ \mathit{Some}.R \circ \neg \quad .$$

(This is just De Morgan's rule.) A simple consequence is that the predicate transformers $\mathit{All}.R \circ \mathit{Some}.R$ and $\mathit{Some}.R \circ \mathit{All}.R$ are also conjugate.

3 Fixed Points

The main significance of the monotonicity properties (13) and (14) is the guaranteed existence of the least and greatest fixed points of compositions of these predicate transformers (where predicates are ordered as usual by "only-if" —i.e. implication everywhere—).

In this section, we first give a very brief summary of fixed-point calculus (subsection 3.1) before motivating a possible relationship between the winning and losing positions in a game, and fixed points of the predicate transformers $\mathit{Some}.R \circ \mathit{All}.R$ and $\mathit{All}.R \circ \mathit{Some}.R$ (subsection 3.2). That these predicate transformers are conjugates leads us to give a brief summary of the properties of fixed points of conjugate (monotonic) predicate transformers (subsection 3.3). The section is concluded by an analysis of moves of different type (subsection 3.4). Taken as a whole, the section establishes strong evidence for the claim that the winning and losing positions are characterised as least fixed points, but does not prove that this is indeed the case.

3.1 Basic Fixed-Point Calculus

We assume that the reader is familiar with fixed-point calculus. (See, for example, [Mat95] for a more detailed account.) The fixed points we consider are of monotonic functions from relations to relations, and from predicates to predicates (so-called predicate transformers). The ordering on relations is the subset ordering —a relation is a set of pairs— , and the ordering on predicates is "only-if" (i.e. implication everywhere).

We use μ to denote the function that maps a monotonic endofunction to its *least* fixed point, and ν to denote the function that maps a monotonic endofunction to its *greatest* fixed point. So, for example, $\mu(\mathit{All}.R \circ \mathit{Some}.R)$ denotes the least fixed point of the predicate transformer $\mathit{All}.R \circ \mathit{Some}.R$, and $\nu(\mathit{All}.R \circ \mathit{Some}.R)$ denotes its greatest fixed point. Sometimes, we need to be explicit about the ordering relation (for example, in the statement of the rolling rule below). If so, we write it as a subscript to μ or ν.

For predicate transformers, the basic rules of the fixed-point calculus are as follows. The least fixed point of the monotonic predicate transformer f is a fixed point of f:

(16) $\quad [\mu f \equiv f.\mu f] \quad ,$

that is "least" (i.e. "strongest") among all prefix points of f: for all predicates p,

(17) $\quad [\mu f \Rightarrow p] \;\Leftarrow\; [f.p \Rightarrow p] \quad .$

The greatest fixed point of f is a fixed point of f:

(18) $\quad [\nu f \equiv f.\nu f] \quad ,$

that is "greatest" (i.e. "weakest") among all postfix points of f: for all predicates p,

(19) $\quad [p \Rightarrow \nu f] \;\Leftarrow\; [p \Rightarrow f.p] \quad .$

Rules (16) and (18) are called the *computation rules*, and rules (17) and (19) are called the *induction rules*.

The *rolling rule* is used several times. Suppose f is a monotonic function to A, ordered by $\leq$, from B, ordered by $\sqsubseteq$, and suppose g is a monotonic function to B from A. Then, $f\circ g$ is a monotonic endofunction on A, and $g\circ f$ is a monotonic endofunction on B. Moreover,

(20) $\quad \mu_{\leq}(f\circ g) \;=\; f\,.\,\mu_{\sqsubseteq}(g\circ f) \quad .$

3.2 Winning and Least Fixed Points

Eliminating $lose$ from (9) and (10), we get:

$$[win \;\Rightarrow\; (Some.M \circ All.M)\,.\,win] \quad .$$

Consequently, by fixed point induction,

$$[win \;\Rightarrow\; \nu(Some.M \circ All.M)] \quad .$$

Note that nowhere does this calculation exploit property (2) —the relation $W \bullet M$ is well-founded— of a winning strategy W. By doing so, we can strengthen the property, replacing "greatest" by "least". The key is to use a fixed-point characterisation of well-foundedness: a relation R is well-founded exactly when the least fixed point of the predicate transformer $All.R$ is everywhere true (see eg. [DBvdW97]). Formally, suppose W is a winning strategy, and suppose p is an arbitrary predicate. We calculate a condition on p, in terms of the predicate transformers $Some.M$ and $All.M$, that guarantees $[dom.W \Rightarrow p]$ as follows.

$[\mathrm{dom}.W \;\Rightarrow\; p]$

$\Leftarrow \quad \{ \quad$ (2) — $W \bullet M$ is well-founded, fixed point induction (17) $\quad \}$

$[\mathrm{All}.(W \bullet M).(\mathrm{dom}.W \Rightarrow p) \;\Rightarrow\; (\mathrm{dom}.W \;\Rightarrow\; p)]$

$= \quad \{ \quad$ aiming to remove leftmost "$(\,\mathrm{dom}.W \Rightarrow)$" , apply (11) $\quad \}$

$[\mathrm{All}.(W \bullet M).(\mathrm{rng}.(W \bullet M) \wedge \mathrm{dom}.W \Rightarrow p) \;\Rightarrow\; (\mathrm{dom}.W \;\Rightarrow\; p)]$

$= \quad \{ \quad$ (3) — in particular, $[\mathrm{rng}.(W \bullet M) \wedge \mathrm{dom}.W \equiv \mathrm{rng}.(W \bullet M)]$ $\quad \}$

$[\mathrm{All}.(W \bullet M).(\mathrm{rng}.(W \bullet M) \Rightarrow p) \;\Rightarrow\; (\mathrm{dom}.W \;\Rightarrow\; p)]$

$= \quad \{ \quad$ (11) $\quad \}$

$[\mathrm{All}.(W \bullet M).p \;\Rightarrow\; (\mathrm{dom}.W \;\Rightarrow\; p)]$

$= \quad \{ \quad$ Leftmost "$(\,\mathrm{dom}.W \Rightarrow)$" has now been removed;

now introduce Some:

for all R, $[\mathrm{dom}.R \equiv \mathrm{Some}.R.\mathrm{true}]$, calculus $\quad \}$

$[\mathrm{All}.(W \bullet M).p \;\wedge\; \mathrm{Some}.W.\mathrm{true} \;\;\Rightarrow\;\; p]$

$= \quad \{ \quad$ All distributes through composition $\quad \}$

$[\mathrm{All}.W.(\mathrm{All}.M.p) \;\wedge\; \mathrm{Some}.W.\mathrm{true} \;\;\Rightarrow\;\; p]$

$\Leftarrow \quad \{ \quad$ for all R, p and q,

$[\mathrm{All}.R.p \wedge \mathrm{Some}.R.q \;\Rightarrow\; \mathrm{Some}.R.(p \wedge q)]$ $\quad \}$

$[\mathrm{Some}.W.(\mathrm{All}.M.p) \;\;\Rightarrow\;\; p]$

$\Leftarrow \quad \{ \quad$ $W \subseteq M$ — (1), Some is monotonic — (12) $\quad \}$

$[(\mathrm{Some}.M \circ \mathrm{All}.M).p \;\Rightarrow\; p]$.

We conclude that $[\mathrm{dom}.W \Rightarrow p]$ if p is any prefix point of $\mathrm{Some}.M \circ \mathrm{All}.M$. Since, $\mu(\mathrm{Some}.M \circ \mathrm{All}.M)$ is the least prefix point of $\mathrm{Some}.M \circ \mathrm{All}.M$, we conclude that

(21) $\quad [\mathit{win} \;\Rightarrow\; \mu(\mathrm{Some}.M \circ \mathrm{All}.M)]$.

A simple calculation gives the corresponding property of lose:

$\mu(\mathrm{All}.M \circ \mathrm{Some}.M)$

$= \quad \{ \quad$ rolling rule: (20) $\quad \}$

$\mathrm{All}\,.\,M\,.\,\mu(\mathrm{Some}.M \circ \mathrm{All}.M)$

$\Leftarrow \quad \{ \quad$ (14) and (21) $\quad \}$

$\mathrm{All}.M.\mathit{win}$

$$= \quad \{ \quad (4), \text{ definition of } \mathit{All} \quad \}$$
$$\mathit{lose} \quad .$$

That is,

(22) $\quad [\mathit{lose} \;\Rightarrow\; \mu(\mathit{All}.M \circ \mathit{Some}.M)] \quad .$

3.3 Conjugate Monotonic Predicate Transformers

We remarked earlier that the predicate transformers $\mathit{Some}.M \circ \mathit{All}.M$ and $\mathit{All}.M \circ \mathit{Some}.M$ are conjugates. In this section, we give a brief summary of fixed-point theory applied to conjugate, monotonic predicate transformers.

Suppose f and g are conjugate, monotonic predicate transformers. Then,

(23) $\quad \neg \circ g = f \circ \neg \quad \wedge \quad \neg \circ f = g \circ \neg \quad .$

Negation is a monotonic function from predicates ordered by "only-if" to predicates ordered by "if". So, by the rolling rule (20) for fixed points,

(24) $\quad [\neg \mu f \;\equiv\; \nu g] \quad .$

An easy consequence of (24) is the following lemma.

Lemma 25 If f and g are conjugate, monotonic predicate transformers, the predicates μf, μg and $\nu f \wedge \nu g$ are mutually distinct:

$$[\neg(\mu f \wedge \mu g) \;\wedge\; \neg(\mu f \wedge (\nu f \wedge \nu g)) \;\wedge\; \neg(\mu g \wedge (\nu f \wedge \nu g))]$$

and together cover all positions:

$$[\mu f \vee \mu g \vee (\nu f \wedge \nu g)] \quad .$$

□

Lemma 26 If f and g are conjugate, monotonic predicate transformers,

$$[\mu f \;\equiv\; \nu f \;\equiv\; \mu f \vee \mu g] \quad .$$

(We remind the reader that continued equivalences should be read associatively, and not conjunctionally [DS90].)

Proof

$$
\begin{array}{ll}
 & \mu f \equiv \nu f \\
= & \{ \quad \text{by definition of } \mu \text{ and } \nu \text{ (“least” and “greatest”)} \\
 & \qquad [\mu f \Rightarrow \nu f] \quad \} \\
 & \mu f \Leftarrow \nu f \\
= & \{ \quad [\neg \mu g \equiv \nu f] \quad \} \\
 & \mu f \Leftarrow \neg \mu g \\
= & \{ \quad \text{predicate calculus} \quad \} \\
 & \mu f \vee \mu g \quad .
\end{array}
$$

□

3.4 Application to Win-Lose Equations

Since $Some.M \circ All.M$ and $All.M \circ Some.M$ are conjugate, monotonic predicate transformers, (21) and lemma 25 suggest that the winning positions are given by the predicate

$$\mu(Some.M \circ All.M) \quad ,$$

the losing positions are given by

$$\mu(All.M \circ Some.M) \quad ,$$

and the stalemate positions are given by

$$\nu(Some.M \circ All.M) \wedge \nu(All.M \circ Some.M) \quad .$$

We abbreviate $\mu(Some.M \circ All.M)$ to $\mathcal{N}$ and $\mu(All.M \circ Some.M)$ to $\mathcal{P}$. We also introduce the predicate $\mathcal{U}$ (short for *unique*) to characterise the positions in which the two fixed-point equations have unique solutions:

$$(27) \qquad [\mathcal{U} \equiv \mathcal{N} \not\equiv \mathcal{P}] \quad .$$

(Note the scope of the everywhere brackets. Note also that “$\equiv$” and “$\not\equiv$” are both associative and symmetric, and associate with each other. This justifies the associative reading of (27). It should *not* be read conjunctionally.)

We abbreviate $\neg\mathcal{U}$ to $\mathcal{O}$. In the standard game-theory nomenclature [BCG82], “$\mathcal{O}$” is short for “open”, “$\mathcal{N}$” is short for “next player wins” and “$\mathcal{P}$” is short for “previous player wins”. We use the nomenclature here *without interpretation* in order to emphasise that we have yet to establish that $\mathcal{N}$ describes the winning positions, $\mathcal{P}$ describes the losing positions, and $\mathcal{O}$ describes the stalemate positions.

By (24) and lemma 26, $\mathcal{U}$ has the property:

(28) $[\mathcal{U} \equiv \mathcal{N} \vee \mathcal{P}]$.

Thus,

(29) $[\mathcal{O} \equiv \neg\mathcal{N} \wedge \neg\mathcal{P}]$.

The following lemma characterises the moves that are possible and not possible between the different types of position. (We use "$\perp\!\!\perp$" to denote the empty relation.)

Lemma 30

(a) From $\mathcal{O}$, there is always a move to $\mathcal{O}$: $[\mathcal{O} \Rightarrow \mathsf{Some.M.}\mathcal{O}]$.

(b) Every move from $\mathcal{O}$ to $\mathcal{U}$ is to $\mathcal{N}$: $\mathcal{O}\bullet\mathsf{M}\bullet\mathcal{U} = \mathcal{O}\bullet\mathsf{M}\bullet\mathcal{N}$.

(c) Moves from $\mathcal{U}$ to $\mathcal{O}$ must start at $\mathcal{N}$: $\mathcal{U}\bullet\mathsf{M}\bullet\mathcal{O} = \mathcal{N}\bullet\mathsf{M}\bullet\mathcal{O}$.

(d) From $\mathcal{N}$, there is always a move to $\mathcal{P}$: $[\mathcal{N} \Rightarrow \mathsf{Some.M.}\mathcal{P}]$.

(e) Every move from $\mathcal{P}$ is to $\mathcal{N}$: $\mathcal{P}\bullet\mathsf{M} = \mathcal{P}\bullet\mathsf{M}\bullet\mathcal{N}$.

(f) Consequently, there are no moves from $\mathcal{P}$ to $\mathcal{O}$: $\mathcal{P}\bullet\mathsf{M}\bullet\mathcal{O} = \perp\!\!\perp$,

(g) and there are no moves between $\mathcal{P}$ and itself: $\mathcal{P}\bullet\mathsf{M}\bullet\mathcal{P} = \perp\!\!\perp$.

(h) Every move to $\mathcal{P}$ is from $\mathcal{N}$: $\mathsf{M}\bullet\mathcal{P} = \mathcal{N}\bullet\mathsf{M}\bullet\mathcal{P}$.

(i) Consequently, there are no moves to $\mathcal{P}$ from $\mathcal{O}$: $\mathcal{O}\bullet\mathsf{M}\bullet\mathcal{P} = \perp\!\!\perp$.

Proof We prove just the first two properties to illustrate the method. The remaining properties are left as exercises. First,

$$
\begin{array}{ll}
 & \mathcal{O} \\
= & \{ \quad \text{definition of } \mathcal{O}\text{: (29)} \quad \} \\
 & \nu(\mathsf{Some.M} \circ \mathsf{All.M}) \wedge \nu(\mathsf{All.M} \circ \mathsf{Some.M}) \\
= & \{ \quad \text{rolling rule: (20) (applied twice)} \quad \} \\
 & \quad \mathsf{Some.M.}\nu(\mathsf{All.M} \circ \mathsf{Some.M}) \\
 & \wedge \ \ \mathsf{All.M.}\nu(\mathsf{Some.M} \circ \mathsf{All.M}) \\
\Rightarrow & \{ \quad \text{for all } \mathsf{R},\ \mathsf{p} \text{ and } \mathsf{q}, \\
 & \quad\quad [\mathsf{All.R.p} \wedge \mathsf{Some.R.q} \ \Rightarrow \ \mathsf{Some.R.(p \wedge q)}] \quad \}
\end{array}
$$

$$
\begin{array}{ll}
& \mathsf{Some}\,.\,\mathsf{M}\,.\,(\nu(\mathsf{All.M}\circ\mathsf{Some.M})\wedge\nu(\mathsf{Some.M}\circ\mathsf{All.M})) \\
= & \{\quad \text{definition of } \mathcal{O}\text{: (29)} \quad\} \\
& \mathsf{Some.M}.\mathcal{O}\ \ .
\end{array}
$$

Second,

$$
\begin{array}{ll}
& \mathcal{O}\bullet\mathsf{M}\bullet\mathcal{U} \\
= & \{\quad [\mathcal{O}\Rightarrow\nu(\mathsf{All.M}\circ\mathsf{Some.M})]\text{, rolling rule: (20)} \quad\} \\
& \mathcal{O}\bullet\mathsf{All.M}.\nu(\mathsf{Some.M}\circ\mathsf{All.M})\bullet\mathsf{M}\bullet\mathcal{U} \\
= & \{\quad \text{for all } p\text{, } \mathsf{All.M}.p\bullet\mathsf{M} = \mathsf{All.M}.p\bullet\mathsf{M}\bullet p \quad\} \\
& \mathcal{O}\bullet\mathsf{All.M}.\nu(\mathsf{Some.M}\circ\mathsf{All.M})\bullet\mathsf{M}\bullet\nu(\mathsf{Some.M}\circ\mathsf{All.M})\bullet\mathcal{U} \\
= & \{\quad [\mathcal{O}\Rightarrow\nu(\mathsf{All.M}\circ\mathsf{Some.M})]\text{,} \\
& \qquad\quad \nu(\mathsf{Some.M}\circ\mathsf{All.M})\bullet\mathcal{U} \\
& \qquad = \quad \{\quad \text{composition of partial identity relations} \\
& \qquad\qquad\qquad \text{is conjunction of predicates} \quad\} \\
& \qquad\quad \nu(\mathsf{Some.M}\circ\mathsf{All.M})\wedge(\mathcal{N}\vee\mathcal{P}) \\
& \qquad = \quad \{\quad [\nu(\mathsf{Some.M}\circ\mathsf{All.M})\wedge\mathcal{N}\equiv\mathcal{N}]\text{,} \\
& \qquad\qquad\qquad [\nu(\mathsf{Some.M}\circ\mathsf{All.M})\wedge\mathcal{P}\equiv\mathsf{false}] \quad\} \\
& \qquad\quad \mathcal{N} \quad\} \\
& \mathcal{O}\bullet\mathsf{M}\bullet\mathcal{N}\ \ .
\end{array}
$$

□

Figure 2 summarises the discussion so far. The three disjoint predicates on positions, $\mathcal{P}$, $\mathcal{N}$ and $\mathcal{O}$, are shown as "clouds". A solid arrow indicates the (definite) existence of a move of a certain type. A dotted arrow indicates the possible existence of a move of a certain type. So, the two solid arrows indicate that, for every $\mathcal{O}$-position, there is a move to an $\mathcal{O}$-position, and, for every $\mathcal{N}$-position, there is a move to a $\mathcal{P}$-position. The dotted arrows indicate that there may be moves from some $\mathcal{O}$-positions to $\mathcal{N}$, from some $\mathcal{N}$-positions to an $\mathcal{O}$-position or an $\mathcal{N}$-position, and from some $\mathcal{P}$-positions to an $\mathcal{N}$-position. Just as important, the absence of arrows indicates the impossibility of certain moves. No moves are possible *from* $\mathcal{P}$ to $\mathcal{O}$ or $\mathcal{P}$, and no moves are possible *to* $\mathcal{P}$ from $\mathcal{O}$ or $\mathcal{P}$.

Suppose that $\mathcal{N}$ does indeed characterise the winning positions. Then, lemma 30 establishes that $\mathcal{O}$ characterises the stalemate positions. After all, from $\mathcal{O}$ it is always possible to remain in $\mathcal{O}$. Moving out of it would place the opponent in an $\mathcal{N}$-position,

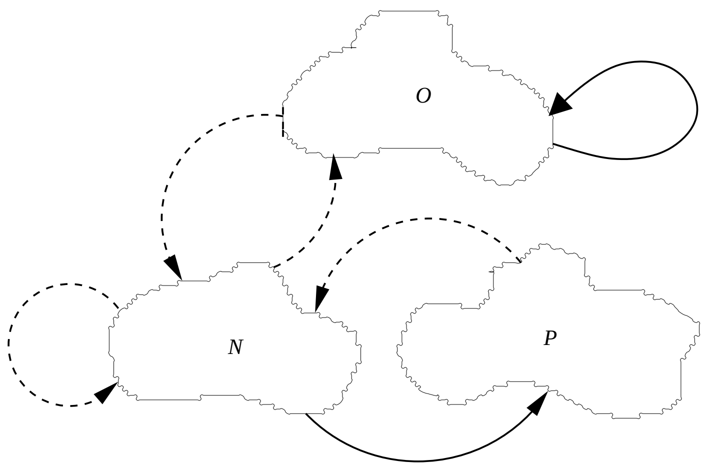

Figure 2: Moves between $\mathcal{P}$-, $\mathcal{N}$- and $\mathcal{O}$-positions. Solid lines mean there *is* a move, dotted lines mean there *may be* a move.

which we have assumed is a winning position. So, from $\mathcal{O}$, the best strategy for both players is to remain in $\mathcal{O}$, waiting for the opponent to make a mistake, and thus continuing the game indefinitely. This is what is meant by "stalemate".

4 Constructing a Winning Strategy

We now turn to the converse of (21). The proof is constructive: we exhibit an algorithm that constructs a winning strategy. Our algorithm is motivated by the following property of the moves from $\mathcal{N}$ to $\mathcal{P}$. (Note that $\mathcal{N}\bullet\mathsf{M}\bullet\mathcal{P}$ is the set of moves from an $\mathcal{N}$-position to a $\mathcal{P}$-position.)

Lemma 31 $\quad [\mathcal{N}\bullet\mathsf{M}\bullet\mathcal{P} \;\equiv\; \langle\mu R :: \mathsf{M}\bullet \mathit{All}.\mathsf{M}.(\mathit{dom}.R)\rangle]$.

Proof

$$
\begin{array}{ll}
 & \mathcal{N}\bullet\mathsf{M}\bullet\mathcal{P} \\
= & \{\quad \mathsf{M}\bullet\mathcal{P} = \mathcal{N}\bullet\mathsf{M}\bullet\mathcal{P}\text{ , definition of } \mathcal{P} \quad\} \\
 & \mathsf{M}\bullet\mu(\mathit{All}.\mathsf{M}\circ \mathit{Some}.\mathsf{M}) \\
= & \{\quad [\mathit{Some}.\mathsf{M}.p \equiv \mathit{dom}.(\mathsf{M}\bullet p)]\text{ , rolling rule: (20)} \quad\} \\
 & \langle\mu R :: \mathsf{M}\bullet \mathit{All}.\mathsf{M}.(\mathit{dom}.R)\rangle \quad .
\end{array}
$$

□

Note that $\mathcal{N}\bullet\mathsf{M}\bullet\mathcal{P}$ is not necessarily a winning strategy. It is in the case that M is well-founded (the case often assumed in texts and papers on game theory). However, the lollipop game (fig. 1) is a simple counterexample: $\mathcal{N}\bullet\mathsf{M}\bullet\mathcal{P}\bullet\mathsf{M}$ is not well-founded because it includes the cycle on position 1.

The fixed-point characterisation suggests how to proceed. We assume that the number of positions is finite and consider an algorithm that incrementally computes a winning strategy W. Initially, W is set to the empty relation (which is easily seen to be a winning strategy). Subsequently, W is augmented by exploiting the fact that positions satisfying $All.\mathsf{M}.(dom.W)$ are losing positions. Specifically, moves from positions not in the domain of W to positions satisfying $All.\mathsf{M}.(dom.W)$ are added to W. Formally, we introduce the function f from relations to relations defined by, for all relations R,

(32) $\quad f.R \;=\; \neg(dom.R)\bullet\mathsf{M}\bullet All.\mathsf{M}.(dom.R)\quad ,$

and the function g defined by, for all relations R,

(33) $\quad g.R \;=\; R\cup f.R\quad .$

Then, the algorithm is as follows:

$\{\;$ number of positions is finite $\;\}$

$\quad W := \perp\!\!\perp$ $\{\;\perp\!\!\perp$ denotes the empty relation $\}$

; $\quad \{\;$ **Invariant:** $WinningStrategy.W\;\}$

$\quad$ do $\;\neg(W = g.W)\;\rightarrow\; W \;:=\; g.W$

$\quad$ od

$\{\;\; WinningStrategy.W \;\wedge\; [dom.W = win = \mathcal{N}]\;\;\}\quad .$

Note that the function g is not monotonic. (Take, for example, M to be the relation $\{(a,b),(a,c)\}$, R to be the empty relation and S to be $\{(a,b)\}$. Then, $R\subseteq S$. But, $g.R = \mathsf{M}$, whereas $g.S = S$. That is, $\neg(g.R\subseteq g.S)$.) So, although the algorithm computes a fixed point of g, standard fixed-point theory does not predict its existence. Nor can the rules of standard fixed-point calculus be used to reason about a fixed point of g.

The assumption that the number of positions is finite gurantees termination because each iteration either adds at least one position to the domain of W or truthifies the termination condition, and there is an upper bound on the number of positions.

In the case that the number of positions is infinite, the algorithm can sometimes be used to determine whether a given position is a winning or losing position, by repeatedly executing the loop body until the position is added to the domain of W (in which case it is a winning position) or to the range of W (in which case it is a losing position). However, examples of games are easily constructed for which some winning positions are not eventually added to the domain of W, so the procedure does *not* constitute a semi-decision procedure for enumerating the winning positions. (The claim made by the authors in the draft paper is thus *wrong*.)

The invariant property is simply that W is a winning strategy. Crucial to establishing the converse of (21) is the claim that, on termination,

$$[dom.W = win = \mathcal{N}] \quad .$$

That is, a maximal winning strategy has been constructed (formally, $[dom.W = win]$) and the winning positions are precisely characterised by the least fixed point of the predicate transformer $Some.M \circ All.M$ (formally, $[win = \mathcal{N}]$).

To establish the conditional correctness of the algorithm, we must verify the invariant property. The invariant is obviously truthified by the assignment $W := \perp\!\!\perp$. To establish that it is maintained by the loop body, it suffices to verify three properties:

(34) $\quad g.W \subseteq M \;\equiv\; W \subseteq M \quad ,$

(35) $\quad WellFounded.(g.W \bullet M) \Leftarrow WinningStrategy.W \quad ,$

(36) $\quad g.W = g.W \bullet All.M.(dom.(g.W)) \;\Leftarrow\; WinningStrategy.W \quad .$

(Compare (34) with (1), (35) with (2), and (36) with (8), the point-free form of (3).)

Property (34) is obviously true. To prove (35), we use the theorem that, for all relations R and S, $R \cup S$ is well-founded if R is well-founded, S is well-founded, and $S \bullet R = \perp\!\!\perp$. (This is a special case of a general theorem on the well-foundedness of the union of well-founded relations [DBvdW97].) So,

$$\begin{array}{ll}
& WellFounded.(g.W \bullet M) \\
\Leftarrow & \{ \quad \text{above mentioned theorem, } g.W = W \cup f.W \quad \} \\
& WellFounded.(W \bullet M) \;\wedge\; WellFounded.(f.W \bullet M) \;\wedge\; W \bullet M \bullet f.W \bullet M = \perp\!\!\perp \\
= & \{ \quad W \bullet M \text{ is well-founded, since } WinningStrategy.W \quad \} \\
& WellFounded.(f.W \bullet M) \;\wedge\; W \bullet M \bullet f.W \bullet M = \perp\!\!\perp \quad .
\end{array}$$

Continuing with each conjunct individually,

$$\begin{array}{ll}
& WellFounded.(f.W \bullet M) \\
= & \{ \quad \text{fixed-point characterisation of well-founded [DBvdW97]} \quad \} \\
& \langle \nu X :: f.W \bullet M \bullet X \rangle = \perp\!\!\perp \\
\Leftarrow & \{ \quad \text{for all } R,\ \langle \nu X :: R \bullet X \rangle = R \bullet R \bullet \langle \nu X :: R \bullet X \rangle
\end{array}$$

$\perp\!\!\perp$ is zero of composition $\}$

$$f.W \bullet M \bullet f.W \;=\; \perp\!\!\perp$$

$\Leftarrow \quad \{$ definition of f, $\perp\!\!\perp$ is zero of composition $\}$

$$All.M.(dom.W) \bullet M \bullet \neg(dom.W) \;=\; \perp\!\!\perp$$

$\Leftarrow \quad \{$ $All.M.(dom.W) \bullet M \;=\; All.M.(dom.W) \bullet M \bullet dom.W$

$\perp\!\!\perp$ is zero of composition $\}$

$$dom.W \bullet \neg(dom.W) \;=\; \perp\!\!\perp$$

$= \quad \{$ composition of predicates is their conjunction $\}$

true .

Similarly,

$$W \bullet M \bullet f.W \bullet M \;=\; \perp\!\!\perp$$

$\Leftarrow \quad \{$ definition of f, $\perp\!\!\perp$ is zero of composition $\}$

$$W \bullet M \bullet \neg(dom.W) \;=\; \perp\!\!\perp$$

$\Leftarrow \quad \{$ by assumption, $WinningStrategy.W$.

So, by (8), $[W \equiv W \bullet All.M.(dom.W)]$.

$\perp\!\!\perp$ is zero of composition $\}$

$$All.M.(dom.W) \bullet M \bullet \neg(dom.W) \;=\; \perp\!\!\perp$$

$= \quad \{$ as above $\}$

true .

Property (36) is proved as follows.

$$g.W \bullet All.M.(dom.(g.W))$$

$\supseteq \quad \{$ $g.W \supseteq W$; dom and All.M are monotonic $\}$

$$g.W \bullet All.M.(dom.W)$$

$= \quad \{$ $g.W = W \cup f.W$, distributivity $\}$

$$W \bullet All.M.(dom.W) \;\cup\; f.W \bullet All.M.(dom.W)$$

$= \quad \{$ by assumption, $WinningStrategy.W$.

So, by (8), $[W \equiv W \bullet All.M.(dom.W)]$.

Also, $f.W = f.W \bullet All.M.(dom.W)$ as composition of partial identity relations is idempotent $\}$

$$
\begin{array}{ll}
 & W \cup f.W \\
= & \{ \quad \text{definition of } g \quad \} \\
 & g.W \quad .
\end{array}
$$

We have thus proved that

$$g.W \bullet \mathit{All}.M.(\mathit{dom}.(g.W)) \supseteq g.W$$

(under the assumption that W is a winning strategy). Equality of the left and right sides (that is (36)) follows from the fact that $\mathit{All}.M.(\mathit{dom}.(g.W))$ is a partial identity relation.

We now turn to the postcondition. We know that $\mathit{WinningStrategy}.W$ is an invariant. So, by definition of a winning strategy, we also know that $[\mathit{dom}.W \Rightarrow \mathit{win}]$ is an invariant. So, on termination, $[\mathit{dom}.W \Rightarrow \mathit{win}]$.

We now show that the converse implication follows from the fact that, on termination, W is a fixed point of g. (Note that g does not have a unique fixed point: $M \bullet \mathcal{P}$ is also a fixed point of g. We leave the proof to the reader.)

We have, for all relations R,

$$
\begin{array}{ll}
 & R = g.R \\
= & \{ \quad \text{set calculus, definition of } g \quad \} \\
 & \neg(\mathit{dom}.R) \bullet M \bullet \mathit{All}.M.(\mathit{dom}.R) \subseteq R \\
\Rightarrow & \{ \quad \mathit{dom} \text{ is monotonic} \quad \} \\
 & [\mathit{dom}.(\neg(\mathit{dom}.R) \bullet M \bullet \mathit{All}.M.(\mathit{dom}.R)) \Rightarrow \mathit{dom}.R] \\
= & \{ \quad \begin{array}{ll} & \mathit{dom}.(\neg(\mathit{dom}.R) \bullet M \bullet \mathit{All}.M.(\mathit{dom}.R)) \\ = & \{ \quad \text{domain calculus} \quad \} \\ & \neg(\mathit{dom}.R) \wedge \mathit{dom}.(M \bullet \mathit{All}.M.(\mathit{dom}.R)) \\ = & \{ \quad [\mathit{dom}.(M \bullet p) \equiv \mathit{Some}.M.p] \quad \} \\ & \neg(\mathit{dom}.R) \wedge (\mathit{Some}.M \circ \mathit{All}.M).(\mathit{dom}.R) \quad \} \end{array} \\
 & [\neg(\mathit{dom}.R) \wedge (\mathit{Some}.M \circ \mathit{All}.M).(\mathit{dom}.R) \Rightarrow \mathit{dom}.R] \\
= & \{ \quad \text{predicate calculus} \quad \} \\
 & [(\mathit{Some}.M \circ \mathit{All}.M).(\mathit{dom}.R) \Rightarrow \mathit{dom}.R] \\
\Rightarrow & \{ \quad \text{fixed-point induction, definition of } \mathcal{N} \quad \} \\
 & [\mathcal{N} \Rightarrow \mathit{dom}.R] \quad .
\end{array}
$$

That is, for all relations R,

(37) $\quad (R = g.R) \;\Rightarrow\; [\mathcal{N} \Rightarrow \mathrm{dom}.R] \quad .$

Combining (37) with (21), we get that, on termination,

$$[(win \Rightarrow \mathcal{N}) \;\wedge\; (\mathcal{N} \Rightarrow \mathrm{dom}.W)] \quad .$$

But, since W is a winning strategy, by definition of win, $[\mathrm{dom}.W \Rightarrow win]$. Thus, by antisymmetry of implication, on termination,

$$[\mathrm{dom}.W = \mathcal{N} = win] \quad .$$

5 Conclusion

There is a very large and growing amount of literature on game theory —too much for us to try to summarise here— and fixed-point characterisations of winning, losing and stalemate positions are likely to be well-known. Nevertheless, the properties we have proved are often justified informally, or simply assumed. For example, Schmidt and Ströhlein [SS93] state lemma 30, but they provide no formal justification. Berlekamp, Conway and Guy [BCG82, chapter 12] assert that every position is in $\mathcal{N}$, $\mathcal{P}$ or $\mathcal{O}$, and describe the algorithm for calculating $\mathcal{P}$-positions. But, their account is entirely informal (in the spirit of the rest of the book, it has to be said).

Surprisingly, exploitation of the simple fact that the predicate transformers $All.M$ and $Some.M$ are conjugate seems to be new; if it is not, then it appears to be not as well-known as it should be. Also, our focus on winning strategies rather than winning positions appears to be novel; we have yet to encounter any publication that formalises the notion of a winning strategy.

However, it would be very wrong to claim that this paper presents novel results. But that is not our goal in writing the paper. Our goal is to use simple games to explain, in a calculational framework, the all-important concepts of least and greatest fixed points to students of computing science. Of course, we would not present this paper, as written here, to our students —concrete examples like those discussed by Berlekamp, Conway and Guy are vital— but it is on the extent to which our paper succeeds in providing a basis for teaching material that we would wish the paper to be judged.

5.0.1 Acknowledgements

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